

Self-similar imploding relativistic shock waves

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Abstract

Self-similar solutions to the problem of a strong imploding relativistic shock wave are calculated. These solutions represent the relativistic generalisation of the Newtonian Gouderley–Landau–Stanyukovich problem of a strong imploding spherical shock wave converging to a centre. The solutions are found assuming that the pre-shocked flow has a uniform density and are accurate for sufficiently large times after the formation of the shock wave.

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I. INTRODUCTION

When a large quantity of energy is injected on a very small volume over an instantaneous time, matter and energy diverge from the deposited volume. This phenomenon generates a very high pressure over the surrounding gas producing an explosion. When the pressure difference between the region of explosion and the matter at rest is sufficiently large, a strong explosion is produced and the discontinuity gives rise to a divergent shock wave. An explosive shock wave has a perfect spherical shape if the volume where the explosion took place is sufficiently small and there are no anisotropies or barriers in the surrounding medium. It was first Sedov [7] who formulated and solved analytically this problem [see also 8]. The problem was also treated by Stanyukovich in his PhD dissertation [see 9, for a full description of the general solution] and in less detail by Taylor [12]. Both authors derived the corresponding self-similar equations and obtained numerical solutions. As shown by Sedov [8], the self-similarity index of the problem is obtained by requiring the energy inside the shock wave to be a fundamental constant parameter. Under these circumstances, the famous Sedov–Taylor similarity index $\alpha = 2/5$ is obtained [5, 13] for a similarity variable $\xi = r/t^\alpha$, where r is the radial distance measured from the point of explosion and t represents time.

The inverse physical situation is represented by a strong imploding spherical shock wave propagating into a uniform density medium. This was first investigated by Guderley [4] and fully solved by Landau & Stanyukovich [see for example 5, 9, 13]. In this case, a detailed construction of the self-similar solution was found by demanding the solution to pass through a singular point admitted by the hydrodynamical equations. The similarity variable in this case is given by $\eta = r/(-t)^\alpha$ with a similarity index $\alpha = 0.68837$ for a monoatomic gas.

The first attempt to build similarity solutions for relativistic hydrodynamics was introduced by the pioneering work of Eltgroth [2, 3] with the identification of a self-similar variable $J := v/c$, where v is the velocity of the flow and c the speed of light. Soon after that, Blandford and McKee [1] solved the relativistic similarity problem for a very strong explosive shock wave. On their analysis, they introduced a similarity variable $\chi = (1 - r/ct)[1 + 2(m+1)\Gamma^2]$, where Γ represents the Lorentz factor of the explosive shock front measured in the rest frame of the un-shocked gas and m is the similarity index. Their analysis represents a relativistic generalisation of the work carried out by Sedov–Taylor–Stanyukovich.

In this article we construct self-similar solutions for the case of a very strong imploding shock wave converging to a centre. This constitutes a relativistic generalisation to the Guderley–Landau–Stanyukovich problem. In order to obtain a unique description of the motion, as it happens for the non-relativistic case, we force the solution to pass through an admissible singularity point, which in turn fixes the value of the similarity index of the flow.

The present article is organised as follows. The next section describes briefly the main equations for relativistic hydrodynamics on a flat space-time and mentions the main results for shock waves on any inertial system of reference. Later in the article, we write down the self-similar equations behind the flow of a strong relativistic imploding shock wave. We calculate the similarity variable, the self-similar index of the problem and solve numerically the equations of motion. Finally, we describe the main properties of the solution.

II. SHOCK WAVES IN RELATIVISTIC HYDRODYNAMICS

The equations that govern the motion of a relativistic flow on a flat space-time are well described by Landau and Lifshitz [5]. First, the conservation of energy and momentum are represented by the condition that the divergence of the energy-momentum tensor $T^{\mu\nu}$ must vanish, i.e.

$$\frac{\partial T^{\mu\nu}}{\partial x^\mu} = 0, \quad (1)$$

with

$$T^{\mu\nu} = wu^\mu u^\nu - p\eta^{\mu\nu}. \quad (2)$$

In here and in what follows Greek indices have values 0, 1, 2, 3 and Latin ones have values 1, 2, 3. As usual, we use Einstein's summation convention over repeated indices. The coordinates x^α are such that $x^\alpha = (ct, \mathbf{r})$, where \mathbf{r} is the spatial radius vector, t represents the time and c is the velocity of light. For a flat space-time, the metric tensor $\eta_{\mu\nu}$ is given by $\eta_{00} = 1$, $\eta_{11} = \eta_{22} = \eta_{33} = -1$ and $\eta_{\alpha\beta} = 0$ for $\alpha \neq \beta$. The four-velocity u^α is defined by the relation $u^\alpha = dx^\alpha/ds$, where the interval ds is given by $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$. All thermodynamical quantities, in particular the pressure p , the enthalpy per unit volume $w = e + p$, the internal energy density e and the temperature T are measured in the proper

frame of the fluid. The internal energy density e constructed in this way contains the rest mass energy density ρc^2 and the “purely” thermodynamical energy density ε , that is

$$e = \rho c^2 + \varepsilon, \quad (3)$$

where ρ represents the proper mass density related to the proper particle number density n and average mass m per particle by the relation

$$\rho = nm. \quad (4)$$

In the absence of sources and sinks of particles, the particle number density n satisfies the continuity equation

$$\frac{\partial n^\alpha}{\partial x^\alpha} = 0, \quad (5)$$

where n^α is the particle number density four-vector defined by $n^\alpha := nu^\alpha$.

For one dimensional flow, equation (1) can be rewritten as two equations, the equation of conservation of energy ($\partial T^{0\alpha}/\partial x^\alpha = 0$) and the equation of conservation of momentum ($\partial T^{1\alpha}/\partial x^\alpha = 0$) respectively as

$$\frac{1}{c} \frac{\partial}{\partial t} \left(\frac{e + \beta^2 p}{1 - \beta^2} \right) + \frac{1}{r^k} \frac{\partial}{\partial r} \left[r^k \beta \frac{p + e}{1 - \beta^2} \right] = 0, \quad (6)$$

$$\frac{1}{c} \frac{\partial}{\partial t} \left(\beta \frac{p + e}{1 - \beta^2} \right) + \frac{1}{r^k} \frac{\partial}{\partial r} \left[r^k \beta^2 \frac{p + e}{1 - \beta^2} \right] + \frac{\partial p}{\partial r} = 0. \quad (7)$$

Planar, cylindrical and spherical symmetric flows are described by these equations when $k = 0, 1$ and 2 respectively.

The relativistic theory of shock waves was first investigated by Taub [10, 11] and is well described by Landau and Lifshitz [5]. Traditionally the analysis is made on a system of reference in which the shock wave is at rest. In a more general approach, McKee and Colgate [6] wrote the necessary equations that describe a shock wave on any inertial system of reference. For the case of an imploding shock wave, it is convenient to choose a system of reference in which the un-shocked gas is at rest. In this particular system of reference the jump conditions across a strong shock wave are simple and take the form [1, 6]

$$\frac{e}{\rho} = \frac{\gamma}{\rho_1} \left[e_1 + p_1 \frac{\beta}{\beta_s} \right], \quad (8)$$

$$\frac{\rho}{\rho_1} = \gamma \frac{\kappa}{\kappa - 1} + \frac{1}{\kappa - 1}, \quad (9)$$

$$\Gamma = \frac{\gamma}{\sqrt{1 - 2(\kappa - 1)/\kappa}}. \quad (10)$$

In the previous equations and in what follows, quantities without subindexes and those with subindex 1 label post-shocked and pre-shocked hydrodynamical quantities respectively. The Lorentz factor of the post-shocked flow and that of the shock wave are represented by γ and Γ respectively. The corresponding values of the velocity measured in units of the speed of light are β and β_s . On both sides of the discontinuity, a Bondi–Wheeler equation of state with constant index κ given by

$$p = (\kappa - 1) e, \quad (11)$$

has been assumed.

III. SELF-SIMILAR EQUATIONS AND BOUNDARY CONDITIONS

Let us consider a strong spherical shock wave converging to a centre. The pre-shocked material is assumed to have a constant density ρ_1 . We analyse the problem in a state such that the radius R of the shock wave is much smaller than the initial radius R_0 it had when it was produced. Under these circumstances, as it happens for the non-relativistic case [5, 9, 13], the flow is largely independent of the specific initial conditions. One can thus imagine a spherical piston that pushes the flow to the centre from very large distances in such a way that it produces a strong shock wave. That is,

$$\frac{p}{n} \gg \frac{p_1}{n_1}. \quad (12)$$

This expression combined with equations (8) and (9) for a gas with the equation of state (11) implies that

$$\gamma - 1 \gg \frac{1}{\kappa - 1} \frac{p_1}{c^2 \rho_1} \left(1 + \frac{\kappa_1}{\kappa_1 - 1} \frac{p_1}{c^2 \rho_1} \right)^{-1}, \quad (13)$$

where κ_1 and κ are the constant indexes for the pre-shocked and post-shocked gas respectively. The Lorentz factor γ for the shocked gas that satisfies the previous inequality is so large that the post-shocked flow velocity is comparable to that of light.

In order to simplify the problem, let us assume that the particles that constitute the shocked-gas are themselves relativistic, so that they obey an ultrarelativistic equation of state with $\kappa = 4/3$ in equation (11). That is,

$$p = \frac{1}{3}e. \quad (14)$$

The shocked gas with this equation of state satisfies the following jump velocity condition

$$\gamma^2 = \frac{1}{2}\Gamma^2, \quad (15)$$

according to equation (10). From this, it is clear that inequality (13) imposes the condition $\Gamma^2 \gg 1$ for a strong shock wave. This justifies the restriction of our further analysis to zeroth order quantities in Γ^2 and γ^2 .

The density jump condition given by equation (9) takes the form

$$\tilde{n} = 4\gamma^2 n_1,$$

in which \tilde{n} represents the particle number density of the shocked gas as measured with respect to the pre-shocked gas, i.e. $\tilde{n} = \gamma n$. In terms of the Lorentz factor Γ of the shock wave, this last equation becomes

$$\tilde{n} = 2\Gamma^2 n_1. \quad (16)$$

Finally, the energy jump condition (8) is given by

$$e = \frac{n}{n_1} \gamma \omega_1 = 2\Gamma^2 \omega_1.$$

In terms of the pressure p , the previous equation is

$$p = \frac{2}{3}\Gamma^2 \omega_1, \quad (17)$$

where $\omega_1 = \rho_1 c^2$ for a cold pre-shocked gas and $\omega_1 = 4p_1$ for an ultrarelativistic one. The three conditions given by equations (15), (16) and (17) represent post-shocked boundary

conditions of the problem. These conditions are exactly the same ones obtained by Blandford and McKee [1] for the case of a relativistic explosive shock wave.

Let us now write the equations of motion for the relativistic flow behind the imploding spherical shock wave. The symmetry of the problem demands that the energy conservation equation (6) and the momentum equation (7) take the form

$$\frac{d}{dt} (p\gamma^4) = \gamma \frac{\partial p}{\partial t}, \quad (18)$$

$$\frac{1}{c} \frac{d}{dt} \ln (p^3 \gamma^4) = - \frac{4}{r^2} \frac{\partial}{\partial r} (r^2 \beta). \quad (19)$$

As a third equation we can either use the continuity equation, or the entropy conservation equation which is obtained from equations (18), (19) and (5). We select here the last one, which for an ultrarelativistic gas is simply given by [5]

$$\frac{d}{dt} \left(\frac{p}{n^{4/3}} \right) = 0. \quad (20)$$

Equations (18), (19) and (20) together with the boundary conditions (15), (16) and (17) completely determine the mathematical problem.

IV. SELF-SIMILAR SOLUTION

To find a self-similar variable for the problem, we recall that for the case of a relativistic strong explosion [1], if R represents the radius of the shock measured from the origin, then the ratio R/Γ^2 is a characteristic parameter of the problem. With this ratio, Blandford and McKee [1] built a similarity variable for the problem. In the case of an imploding shock wave, none of the arguments in favour of the characteristic ratio R/Γ^2 are valid and instead we must construct the similarity variable by other means. The Appendix shows an alternative way in which the similarity variable obtained by Blandford and McKee can be calculated. The advantage of this approach is that we can generalise the method for the case we are dealing with in this article.

The similarity variable for the case of a strong relativistic implosion is found as follows. The radius of the shock wave R converges to the origin as time increases, so that its velocity

dR/dt must be negative. Thus,

$$\frac{1}{c} \frac{dR}{dt} = - \left[1 - \frac{1}{\Gamma^2} \right]^{1/2} \approx - \left[1 - \frac{1}{2\Gamma^2} \right], \quad (21)$$

In order to integrate this equation, we must know the time dependence of the Lorentz factor Γ . For the case of a strong relativistic explosion, this is given by equation (A3). We can use a similar time power law dependence by taking into consideration the fact that the shock wave accelerates as it converges to a point. In other words, the Lorentz factor Γ must increase as time advances, that is

$$\Gamma^2 = A (-t)^{-m}, \quad (22)$$

for times $t \leq 0$. The value of the constant factor A is determined by the specific initial conditions of the problem and m is the similarity index. The time interval $t \leq 0$ has been chosen in complete analogy to the Newtonian case [cf. 5, 13]. Under these circumstances, the time $t = 0$ corresponds to a time in which the shock wave collapses to the origin, i.e. $R = 0$.

Substitution of equation (22) on relation (21) yields the integration

$$R = c |t| \left[1 - \frac{1}{2(m+1)\Gamma^2} \right]. \quad (23)$$

In exactly the same form in which we built the similarity variable for the case of a strong relativistic explosion (cf. Appendix), let us choose here the similarity variable η as

$$\begin{aligned} \eta &= 1 + 2(m+1)\Gamma^2 (1 - r/R), \\ &= 1 + 2(m+1)\Gamma^2 \left[1 + \frac{r}{ct} \frac{1}{(1 - 1/2(m+1)\Gamma^2)} \right]. \end{aligned}$$

Thus, to order $O(\Gamma^{-2})$, this similarity variable takes the form

$$\eta = \left(1 + \frac{r}{ct} \right) [1 + 2(m+1)\Gamma^2]. \quad (24)$$

The problem we are concerned is such that the relevant region $r \geq R$ corresponds to the values $1 \geq \eta \geq -\infty$ of the similarity variable η .

For this kind of self-similar description, Buckingham's theorem of dimensional analysis [8] motivated by the boundary conditions given by equations (15), (16) and (17), demands the existence of three dimensionless hydrodynamical functions $f(\eta)$, $g(\eta)$ and $h(\eta)$ given by

$$p = \frac{2}{3}\omega_1\Gamma^2 f(\eta), \quad (25)$$

$$\tilde{n} = 2n_1\Gamma^2 h(\eta), \quad (26)$$

$$\gamma^2 = \frac{1}{2}\Gamma^2 g(\eta). \quad (27)$$

These dimensionless functions should be analytical and satisfy the following boundary condition at the shock wave

$$f(\eta = 1) = g(\eta = 1) = h(\eta = 1) = 1. \quad (28)$$

The introduction of the similarity variable η means that the hydrodynamical functions change their independent variables to Γ^2 and η , instead of r and t . The derivatives of r and t in terms of these new variables are thus given by

$$\frac{\partial}{\partial \ln t} = -m \frac{\partial}{\partial \ln \Gamma^2} + [1 + (m+1)(2\Gamma^2 - \eta)] \frac{\partial}{\partial \eta}, \quad (29)$$

$$ct \frac{\partial}{\partial r} = [1 + 2(m+1)\Gamma^2] \frac{\partial}{\partial \eta}. \quad (30)$$

Using these two equations, the total time derivative $d/dt = \partial/\partial t + v\partial/\partial r$ takes the form

$$\frac{d}{d \ln t} = -m \frac{\partial}{\partial \ln \Gamma^2} + (m+1) \left(\frac{2}{g} - \eta \right) \frac{\partial}{\partial \eta}. \quad (31)$$

With the aid of equations (25), (26) and (27) we can rewrite equations (18) and (19) in terms of Γ^2 and η . In fact, to $O(\Gamma^{-2})$ we obtain

$$-(m+1)(2+g\eta) \frac{f'}{fg} + 4(m+1)(2-g\eta) \frac{g'}{g^2} = 3m, \quad (32)$$

$$\frac{f'}{fg} [(m+1)(8-2g\eta)] - 8(m+1) \frac{g'}{g^2} = 2m-8, \quad (33)$$

where the prime denotes derivative with respect to η . These two equations can be rewritten in matrix form as

$$\begin{bmatrix} -(2 + g\eta) & 2(2 - g\eta) \\ 8 - 2g\eta & -8 \end{bmatrix} \begin{bmatrix} f'/fg \\ g'/g^2 \end{bmatrix} = \begin{bmatrix} 3m/(m+1) \\ (2m-8)/(m+1) \end{bmatrix}, \quad (34)$$

which implies, according to Cramer's rule, that

$$\frac{1}{fg} \frac{df}{d\eta} = \frac{8(m-1) + g\eta(4-m)}{(m+1)[-4 + 8g\eta - g^2\eta^2]}, \quad (35)$$

$$\frac{1}{g^2} \frac{dg}{d\eta} = \frac{4 - 7m + g\eta(2+m)}{(m+1)[-4 + 8g\eta - g^2\eta^2]}. \quad (36)$$

Equations (35) and (36) are accompanied by the boundary conditions (28) for $f(\eta)$ and $g(\eta)$.

The unique value for the similarity index m and the uniqueness of the solution are determined by the following physical consideration, analogous to the one presented in the Newtonian case [cf. 9].

For a given time, the values of the energy density and velocity of a particular fluid element must decrease as the radius r increases. For the case of an ultrarelativistic gas, these quantities can be written in terms of the pressure p and the Lorentz factor γ of the flow respectively. Thus, a necessary condition for the flow that guarantees the decrease of these quantities is given by

$$\frac{\partial p}{\partial r} < 0, \quad \frac{\partial \gamma^2}{\partial r} < 0, \quad \forall r \geq R.$$

In terms of dimensionless quantities, the previous conditions are

$$\frac{df(\eta)}{d\eta} > 0, \quad \frac{dg(\eta)}{d\eta} > 0, \quad \forall \eta \leq 1. \quad (37)$$

The integral functions $f(\eta)$ and $g(\eta)$ depend on the choice of the parameter m and so, a family of integral curves can be found for different self-similar flows. However, with the restrictions imposed by inequalities (37), it is possible to integrate the differential equations (35) and (36) in a unique form. This happens because the conditions (37) impose a single value for the similarity index m .

The derivative $dg(\eta)/d\eta$ in equation (36) has a positive value at the surface of the shock wave, i.e. when $\eta = g = 1$. In order to guarantee a positive value for all $\eta < 1$, $g(\eta)$

must avoid extrema. In the same form, the continuity of the function $g(\eta)$ means that the denominator on the right hand side of equation (36) should not vanish. To overcome these problems, we restrict the integral curve to pass through the singular point $(\eta^*, g(\eta^*))$ for which the numerator and denominator on equation (36) vanish simultaneously.

The denominator on equation (36) vanishes when the following condition is satisfied

$$g\eta = 4 \pm 2\sqrt{3}, \quad (38)$$

which represents a hyperbola on the $g\eta$ plane. The numerator of the same equation has a null value on the hyperbola

$$4 - 7m + g\eta(2 + m) = 0, \quad (39)$$

The point $(\eta^*, g(\eta^*))$ where both hyperbolas intersect is calculated by direct substitution of the value $g\eta$ on equation (39), choosing the negative sign so that the number m remains positive. This fixes a unique value for the similarity index m given by

$$m = 12\sqrt{3} - 20 = 0.78460969. \quad (40)$$

If we now follow the same procedure for the derivative $df(\eta)/d\eta$ in equation (35), we obtain the same value for the similarity index m as the one given by the previous equation.

The value of the similarity index given by equation (40) guarantees that pressure and velocity gradients have negative values for the shocked gas. Using a 4th rank Runge–Kutta method we have calculated the integral curves for $g(\eta)$ and for $f(\eta)$. These results are plotted in Fig. 1.

Let us now obtain a differential equation for the function $h(\eta)$. This is obtained by direct substitution of the differential equations (35) and (36) in equation (20), that is

$$\frac{1}{gh} \frac{dh}{d\eta} = \frac{2(8 - 9m) + 2g\eta(5m - 6) - g^2\eta^2(m - 2)}{(2 - g\eta)(m + 1)[-4 + 8g\eta - g^2\eta^2]}. \quad (41)$$

The integral curve $h(\eta)$ of this equation shown in Figure 1 was calculated with the already integrated values of $g(\eta)$ and the unique value of m given by equation (40). The number m is such that when the numerator on the right hand side of the previous equation vanishes, the numerator also does so, avoiding singularities and extrema in the range $\eta \leq 1$. Furthermore, the derivative $dh/d\eta$ evaluated on $\eta = 1$ has a negative value. Thus, the particle number

density \tilde{n} as presented by relation (26) grows monotonically as r increases for fixed values of time t , i.e.

$$\frac{dh(\eta)}{d\eta} < 0. \quad (42)$$

This peculiar situation also occurs for the non-relativistic imploding shock wave and will be discussed in more detail in the next section.

With the integral functions $f(\eta)$, $g(\eta)$ and $h(\eta)$ calculated in this section it is then possible to calculate the dimensional hydrodynamical quantities using equations (25), (26) and (27).

V. DISCUSSION

The self-similar solution found in this article shows that some of the values of the hydrodynamical quantities decrease in a given fluid element as it moves behind the shock wave. For example, the variation of the pressure in time as a fluid element moves behind the shock wave satisfies the following inequality

$$\begin{aligned} \frac{dp}{dt} &= -\frac{1}{|t|} \frac{dp}{d \ln t}, \\ &= \frac{1}{|t|} \left\{ -m \frac{\partial}{\partial \ln \Gamma^2} + (m+1) \left(\frac{2}{g} - \eta \right) \frac{\partial}{\partial \eta} \right\} p, \\ &= -\frac{1}{|t|} \left\{ -mp + p(m+1)(2 - g\eta) \left[\frac{1}{gf} \frac{\partial f}{\partial \eta} \right] \right\}, \\ &< -\frac{1}{|t|} \left\{ -(m+1)p + p(m+1)(2 - g\eta) \left[\frac{1}{gf} \frac{\partial f}{\partial \eta} \right] \right\}, \\ &< -\frac{1}{|t|} \left\{ p(m+1)(1 - g\eta) \left[\frac{1}{gf} \frac{\partial f}{\partial \eta} \right] \right\}. \end{aligned} \quad (43)$$

The solution obtained in the previous section satisfies the condition $g\eta \leq 1$, for $\eta \leq 1$ (cf. Figure 1). From this and the condition in equation (37), it follows that the total time derivative dp/dt has a negative value for any fluid element behind the shock wave. Using similar arguments it is easy to prove that, as the shocked gas particles move away from the shock wave, the Lorentz factor γ decreases and the particle number density \tilde{n} increases. These results combined with the fact that $\tilde{n} = \gamma n$ imply that the proper particle number density n increases as the fluid particles move away from the shock wave, i.e. $dn/dt > 0$. This result is exactly what is obtained in the non-relativistic case [13]. The particle number

density increases behind the shock and at sufficiently large distances behind the shock it must converge to a finite value.

Self-similar solutions of the second type (see e.g. Zel'dovich and Raizer [13]) such as the one found in this article satisfy two basic properties. The first one is the fact that the integral curve passes through a singular point (for a specific value of the similarity index). The second one is the existence of a curve $r(t)$ on the r - t plane which corresponds to the singular point η_* and is itself a C_- characteristic bounding the region of influence. In order to verify this last statement for the self-similar solution found above, we proceed as follows. Let $\eta_* = \text{const.}$ correspond to a value of the similarity variable evaluated at the singular point. Using equations (22) and (24) it is then straightforward to calculate the differential equation satisfied by the curve $r_*(t)$ evaluated at the singular point:

$$\frac{dr_*}{dt} = -c + \frac{\eta_* c}{2\Gamma^2}, \quad (44)$$

at $O(\Gamma^{-2})$. On the other hand, at the singular point, equation (36) implies that the product $\eta_* g_* = 2/(2 + \sqrt{3})$ for the case $0 \leq \eta \leq 1$ which corresponds to the light causal region $R \leq r \leq c|t|$, according to equations (23) and (24). With this, equation (44) can be rewritten as

$$\frac{dr_*}{dt} = -c + \frac{c}{2(2 + \sqrt{3})\gamma_*^2}. \quad (45)$$

The curve $r(t)$ that describes the trajectory of any C_- characteristic on the r - t plane is described by the following differential equation [10, 11]

$$\frac{dr}{dt} = -c \frac{\alpha - \beta}{1 - \alpha\beta}, \quad (46)$$

where α and β are the sound speed and the velocity of the flow measured in units of the speed of light c respectively. For the self-similar solution we are dealing with in this article, $\beta \approx -1 + 1/2\gamma^2$. This implies that at $O(\gamma^{-2})$, equation (46) takes the following form

$$\frac{dr}{dt} = -c + \frac{c(1 - \alpha)}{2\gamma^2(1 + \alpha)}. \quad (47)$$

Now, using the equation of state (14), it follows that the value of the velocity of sound in units of the speed of light is given by $\alpha = 1/\sqrt{3}$. So, equation (47) has exactly the same form as equation (45). This means that the curve $r_*(t)$ that corresponds to the singular value η_*

is itself a C_- characteristic. In what follows we label this particular characteristic as C_-^* . From the above results, an important condition about the causality of the flow behind the imploding shock wave is obtained. Since the curve $r_*(t)$ that corresponds to the singular point η_* does not intersect the imploding shock wave (except at the origin of the $r-t$ plane) and because the C_- characteristics do not intersect one another, there is a region near the shock wave for which all C_- characteristics intersect the shock wave. Beyond the curve $r_*(t)$, the C_- characteristics do not intersect the imploding shock wave. This means that the flow farther away from the singular C_-^* characteristic does not have any causal influence on the shock wave. These physical consequences on the causality are exactly the same as the ones that occur for the non-relativistic imploding shock wave.

The energy contained inside a gas shell that moves in a self-similar manner can be calculated as follows. First, let us write down the value for the energy E contained on a shell of shocked gas. For an ultrarelativistic gas, the energy density T^{00} measured on the system of reference in which the pre-shocked gas is at rest is given by $p(4\gamma^2 - 1)$. Thus, to $O(\gamma^{-2})$ inside the shell limited by the radius $R_0(t)$ and $R_1(t)$ the energy is given by

$$E(R_0(t), R_1(t), t) = 16\pi \int_{R_0}^{R_1} p\gamma^2 r^2 dr. \quad (48)$$

When the limits of integration are taken from the radii $r = R$ to $r_* = c|t|(1 - \eta_*/[1 + 2(m+1)\Gamma^2])$, i.e. the self-similar region of the flow, we obtain

$$\begin{aligned} E &= 8\pi\omega_1\Gamma^4 |t|^3 (1 + 2(m+1)\Gamma^2)^{-1} \int_1^{\eta_*} fg \left(1 - \frac{\eta}{1 + 2(m+1)\Gamma^2}\right)^2 d\eta, \\ &\approx 8\pi\omega_1\Gamma^2 |t|^3 \Xi(\eta_*). \end{aligned} \quad (49)$$

In these relations, the quantity $\Xi(\eta_*)$ is a constant and all quantities were approximated to $O(\Gamma^{-2})$. Substitution of equation (22) in equation (49) yields

$$E = 8\pi\omega_1 A |t|^{3-m} \Xi(\eta_*). \quad (50)$$

This equation, together with the value of the similarity index given by equation (40), means that the energy contained inside the shell is proportional to a positive power of time. In other words, the energy inside this self-similar shell tends to zero as the imploding shock wave collapses to the origin, which occurs when $t \rightarrow 0$. As the shock wave converges to a point,

energy becomes concentrated right behind the shock front. However the dimensions of the self-similar region decrease and the previous analysis shows that the energy concentrated within this region also decreases.

The constant A was introduced in equation (22). Its value is determined by the initial conditions that generate the shock wave, in complete analogy with the Newtonian case (cf. [13]). Thus, through a specified value of this constant, different flows behind the imploding shock wave are generated for different values of the initial pressure discontinuity. Figures 2 to 4 show examples of numerical integrations of the equations of motion. These solutions were calculated in natural units such that $c = k_B = 1$, with k_B representing Boltzmann's constant. The pre-shocked uniform medium was normalised in order to have a particle number density $n_1 = 1$, initial pressure $p_1 = 1$ and initial enthalpy per unit volume $w_1 = mn_1c^2 = 1$. These assumptions fix uniquely the units of mass, length and time for the specific problem. The initial pressure discontinuity that gave rise to the shock was assumed to have a value $p_2 = 100$ at a distance $R_0 = 100$. The approximations made in this article break down for values $\gamma \lesssim 10$. As discussed before, the pressure and the Lorentz factor behind the flow decrease with increasing radius. The particle number density seems to grow without limit behind the shock wave. This unphysical behaviour occurs because the solutions discussed above are valid only for $O(\gamma^{-2})$, and the region where the proper particle number density converge does not satisfy this approximation. Only a full self-similar solution of the problem, with no first order approximations in γ^{-2} , will show that behaviour.

For the particular case mentioned in this article, we have presented the last stages before the collapse of an imploding shock wave, which was generated by a very strong external pressure and where the complete treatment requires a relativistic analysis. At these final stages of the implosion, the flow pattern is not too sensitive of the specific initial conditions of the problem.

Finally, we mention the regime under which the discussed solution remains valid. This is simply determined by the value of the post-shocked flow velocity. For sufficiently large distances behind the shock wave, the velocity of the gas β in units of the speed of light reduces so that its Lorentz factor $\gamma \sim 1$. In this velocity regime, the approximations made on this article are no longer valid and the relativistic self-similar solution has no meaning.

Acknowledgments

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APPENDIX A: RELATIVISTIC SIMILARITY VARIABLE FOR A STRONG EXPLOSION

Let us consider a relativistic strong explosion, diverging from the origin of coordinates. As mentioned in the article, similarity solutions were found by Blandford and McKee [1]. Let us show in here how to obtain their similarity variable with an alternative argument that can be extended naturally for the case of a strong relativistic implosion.

The rate of change of the radius $R(t)$ of the shock wave in units of the speed of light is given by

$$\frac{1}{c} \frac{dR}{dt} = \left(1 - \frac{1}{\Gamma^2}\right)^{1/2}, \quad (\text{A1})$$

where Γ represents the Lorentz factor of the shock wave. A Taylor expansion of the right hand side of this equation to $O(\Gamma^{-2})$ gives

$$\frac{dR}{dt} = c \left(1 - \frac{1}{2\Gamma^2}\right). \quad (\text{A2})$$

Let us consider that Γ^2 is a power law function of time only, so that

$$\Gamma^2 = \frac{A}{t^m}, \quad (\text{A3})$$

in which A is an unknown constant (but can be found from the specific initial conditions of the problem) and the constant m represents the similarity index. Substitution of equation (A3) on (A2) yields the integral

$$R = ct \left(1 - \frac{1}{m+1} \frac{1}{2\Gamma^2}\right). \quad (\text{A4})$$

The product ct has dimensions of length. Thus, we can write the dimensionless ratio

$$\frac{R}{r} := 1 - \frac{\phi}{2\Gamma^2(m+1)},$$

where we have introduced a new dimensionless quantity ϕ . To first order approximation, the inverse of the previous relation is given by

$$\frac{r}{R} = 1 + \frac{\phi}{2\Gamma^2(m+1)},$$

so that the dimensionless variable ϕ is

$$\phi = \left(1 - \frac{r}{R}\right) 2(m+1)\Gamma^2, \quad (\text{A5})$$

and vanishes when $r = R$.

Self-similar motion means that all hydrodynamical functions can be written as the product of two dimensionless functions $\Pi(t)$ and $\pi(r/R)$ [8, 13], which is satisfied by the parameter ϕ on equation (A5). To avoid a null value of this variable at the shock radius, it is more convenient to take as similarity variable the function $\chi := \phi + 1$, and not ϕ , i.e.

$$\chi = 1 + \left(1 - \frac{r}{R}\right) 2(m+1)\Gamma^2. \quad (\text{A6})$$

The range of this similarity variable should be taken for $\chi \geq 1$. The value $\chi = 1$ corresponds to the shock wave. As described by Blandford and McKee [1], the hydrodynamical equations simplify to great extent when this variable is used. Substitution of equations (A3) and (A4) on (A6) yields an expression for χ at $\mathcal{O}(\Gamma^{-2})$ given by

$$\chi = \left(1 - \frac{r}{ct}\right) [1 + 2(m+1)\Gamma^2]. \quad (\text{A7})$$

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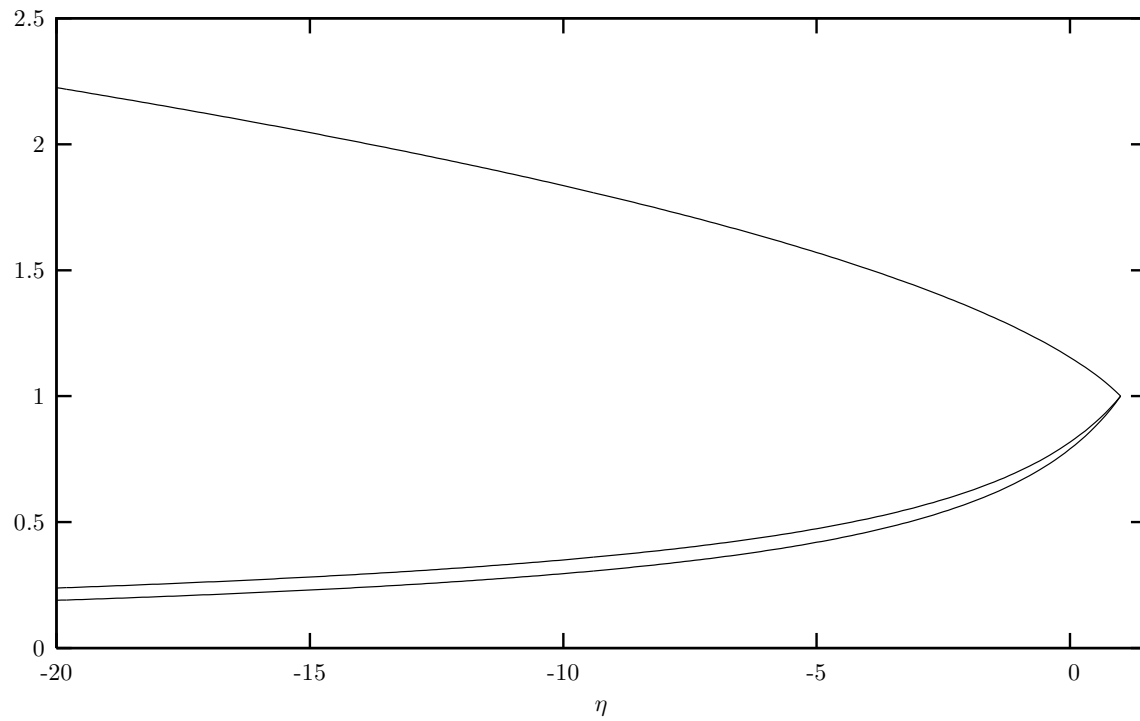


FIG. 1: The figure shows, from bottom to top, the integral curves of the dimensionless pressure $f(\eta)$, squared Lorentz factor $g(\eta)$ and particle number density $h(\eta)$ as functions of the similarity variable η .

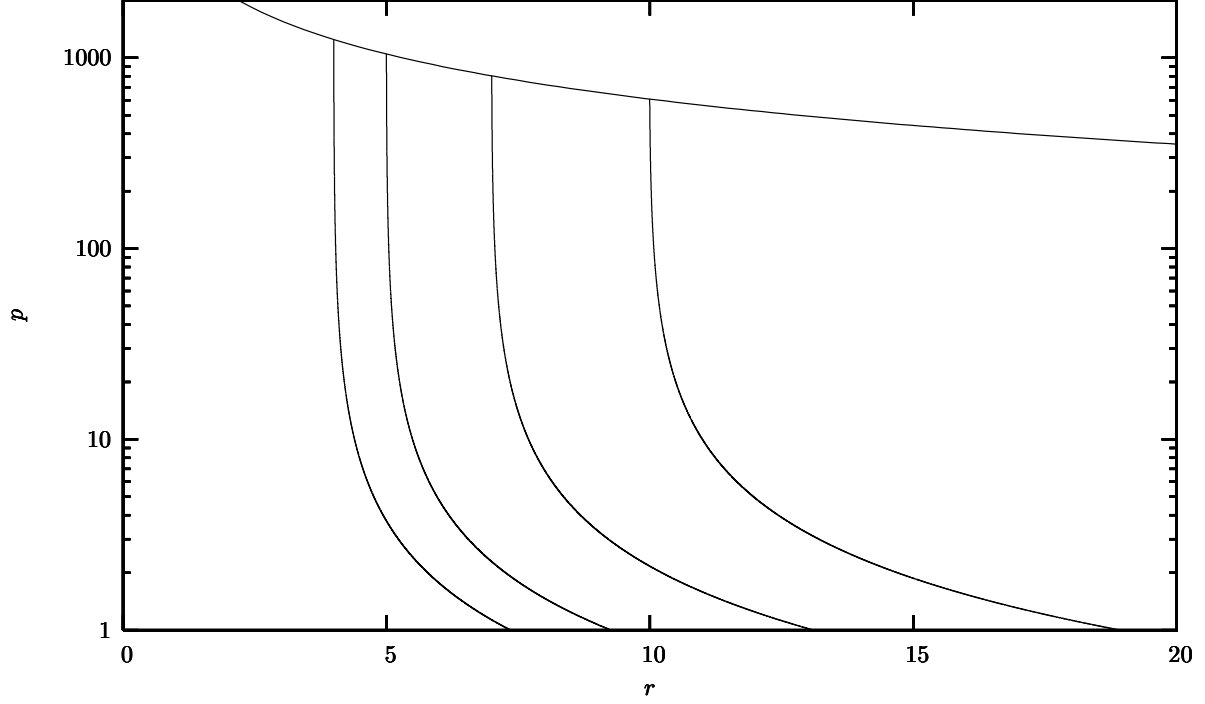


FIG. 2: The figure shows pressure p profiles behind a strong relativistic imploding shock wave. Natural units for the plot were used. The ambient medium was assumed to be of uniform density, with a particle number density $n_1 = 1$, a pressure $p_1 = 1$ and an enthalpy per unit volume $w_1 = 1$. The imploding shock wave was considered to be formed at a radius $r_0 = 100$, where the initial pressure discontinuity $p_2 = 100$. From left to right, the figure shows the pressure distribution as a function of radius r for time values of 4, 5, 7 and 10 before the collapse of the shock wave. The envelope for the different values of the pressure just behind the shock wave is also shown. The pressure grows without limit as the shock wave converges to the origin.

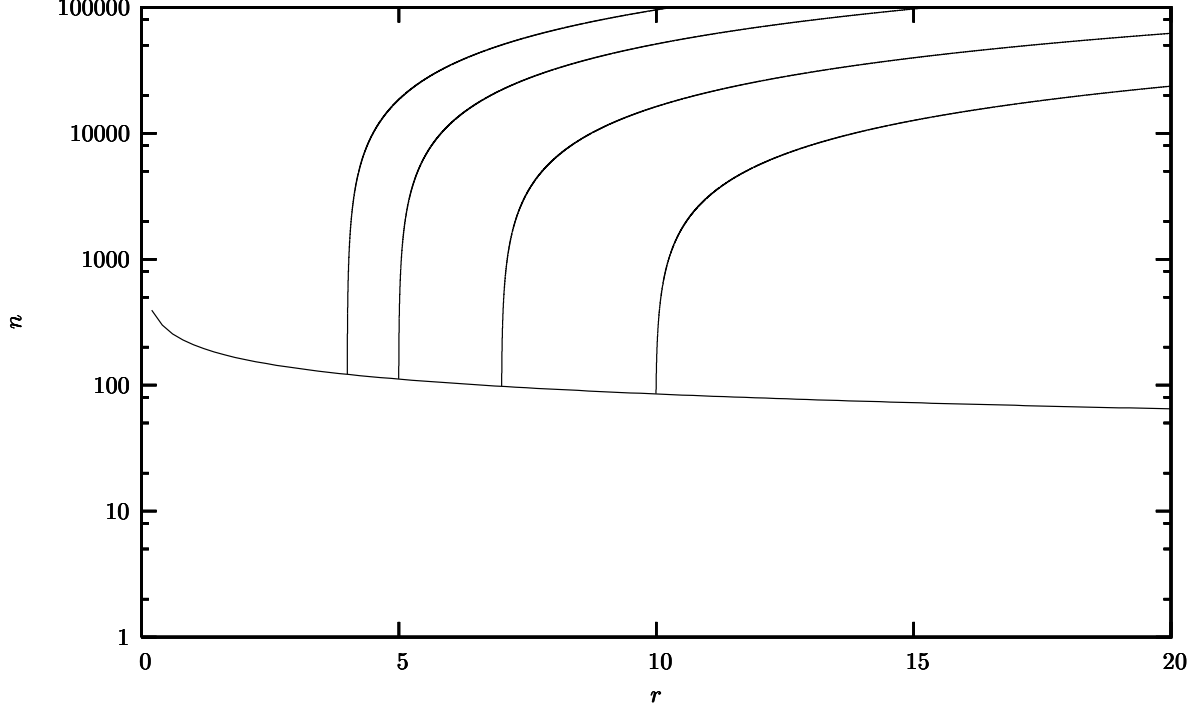


FIG. 3: Particle number density n profiles for the conditions mentioned in Figure 2 are shown in the figure, as a function of the radial distance r . From left to right, the plotted curves show the particle number density distribution for times of 4, 5, 7 and 10 previous to the collapse of the shock wave respectively. The envelope curve shows the particle number density for the gas that flows just behind the shock wave. The particle number density n tends to infinity as the shock wave reaches the origin. Note that the particle number density profiles appear to grow without limit for sufficiently large radius behind the shock wave. This is a false result due to the fact that the Lorentz factor of the flow γ is such that $\gamma^2 \lesssim 10$ and the assumptions made on this article are no longer valid on this regime (see text for more information on this).

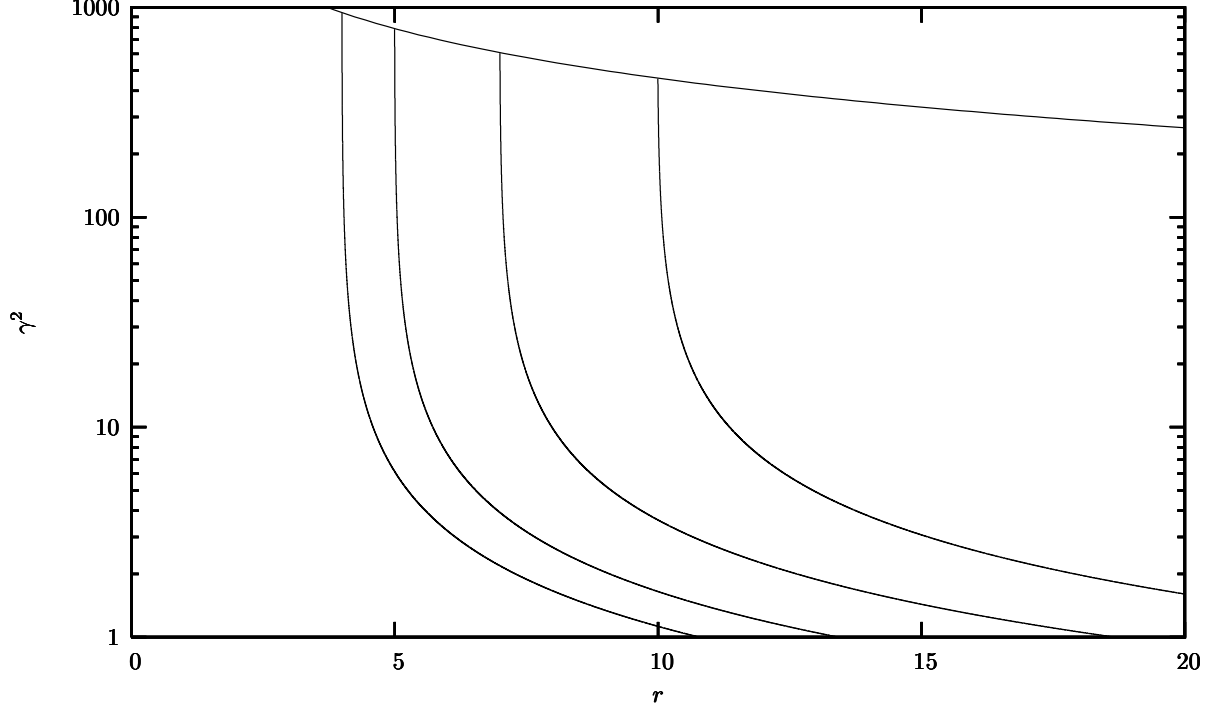


FIG. 4: Different Lorentz factor γ profiles as function of the radial distance r are shown in the figure. The profiles are calculated for the shocked gas behind a strong imploding relativistic shock wave travelling through a constant gas medium with the same conditions as the one shown in Figure 2. This plot shows, from left to right, Lorentz factor values corresponding to times of 4, 5 7 and 10 before the collapse of the imploding shock wave. The envelope curve shows the values of the Lorentz factor for particles that have just crossed the strong shock wave. As the plot shows, the Lorentz factor diverges as the shock wave collapses to the origin. Note that the solution breaks down for sufficiently small values of γ such that the $O(\gamma^{-2})$ approximation is no longer valid.